Electromotive force for an anisotropic turbulence: Intermediate nonlinearity

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A nonlinear electromotive force for an anisotropic turbulence in the case of intermediate nonlinearity is derived. The intermediate nonlinearity implies that the mean magnetic field is not strong enough to affect the correlation time of a turbulent velocity field. The nonlinear mean-field dependencies of the hydrodynamic and magnetic parts of the α effect, turbulent diffusion, and turbulent diamagnetic and paramagnetic velocities for an anisotropic turbulence are found. It is shown that the nonlinear turbulent diamagnetic and paramagnetic velocities are determined by both an inhomogeneity of the turbulence and an inhomogeneity of the mean magnetic field **B**. The latter implies that there are additional terms in the turbulent diamagnetic and paramagnetic velocities $\propto \nabla B^2$ and $\propto (\mathbf{B}\cdot\nabla)\mathbf{B}$. These effects are caused by a tangling of a nonuniform mean magnetic field by hydrodynamic fluctuations. This increases the inhomogeneity of the mean magnetic field. It is also shown that in an isotropic turbulence the mean magnetic field causes an anisotropy of the nonlinear turbulent diffusion. Two types of nonlinearities in magnetic dynamo determined by algebraic and differential equations are discussed. Nonlinear systems of equations for axisymmetric $\alpha\Omega$ dynamos in both spherical and cylindrical coordinates are derived.

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I. INTRODUCTION

Turbulent motions of a conducting fluid can generate large-scale (mean) magnetic field and small-scale magnetic fluctuations. Many dynamo models (see, e.g., Refs. [1–5]) are kinematic (i.e., they predict a magnetic field that grows without limit). In order to find, e.g., the magnitude of the magnetic field, the nonlinear effects which limit the field growth must be taken into account. The nonlinearities in turbulent mean-field dynamo imply an effect of a mean magnetic field on the α effect, turbulent magnetic diffusion, turbulent diamagnetic velocity, etc. The mean magnetic field **B** is determined by an induction equation

$$\partial \mathbf{B}/\partial t = \nabla \times [\mathbf{V}\times\mathbf{B} + \boldsymbol{\mathcal{E}}] + \eta\Delta\mathbf{B} \tag{1}$$

(see, e.g., Refs. [1–5]) where **V** is a mean velocity (e.g., the differential rotation), η is the magnetic diffusion due to the electrical conductivity of fluid, $\boldsymbol{\mathcal{E}} = \langle \mathbf{u}\times\mathbf{h}\rangle$ is the turbulent electromotive force, **u** and **h** are fluctuations of the velocity and magnetic field, respectively, angular brackets denote averaging over an ensemble of turbulent fluctuations. The turbulent electromotive force in kinematic dynamo for an isotropic turbulence is given by

$$\boldsymbol{\mathcal{E}} = \alpha_0^{(v)}\mathbf{B} + \mathbf{U}_0\times\mathbf{B} - \eta_T \nabla\times\mathbf{B} \tag{2}$$

(see Ref. [3]), where $\alpha_0^{(v)} = -(1/3)\langle \tau\mathbf{u}\cdot(\nabla\times\mathbf{u})\rangle$ is the hydrodynamic part of the α effect, $\mathbf{U}_0 = -(1/2)\nabla\langle \tau\mathbf{u}^2\rangle$ is the turbulent diamagnetic velocity, $\eta_T = (1/3)\langle \tau\mathbf{u}^2\rangle$ is the turbulent magnetic diffusion, and τ is the correlation time of the turbulent velocity field.

In the nonlinear stage of evolution of the mean magnetic field, the α effect, turbulent diffusion and turbulent diamagnetic velocity depend on the mean magnetic field **B**. The total α effect in nonlinear dynamo is split into hydrodynamic $\alpha^{(v)}$ and magnetic $\alpha^{(h)}$ parts, where $\alpha^{(h)} = (\tau/3\mu_0\rho)\langle \tau\mathbf{h}\cdot(\nabla\times\mathbf{h})\rangle$ and μ_0 is the magnetic permeability of the fluid (see Refs. [6–8]). Such splitting of the α effect is introduced in nonlinear dynamo because the growing magnetic field reacts differently on the hydrodynamic and the magnetic parts of the α effect (see Refs. [9–11]). The back reaction of the mean magnetic field on the hydrodynamic part of the α effect is almost instantaneous (of the order of a characteristic correlation time of the turbulence, $\tau_0 = l_0/u_0$, where u_0 is the characteristic turbulent velocity in the maximum scale of turbulent motions l_0). However, the characteristic time τ_h of the back action of the mean magnetic field on the magnetic part of the α effect is much larger than τ_0 for large magnetic Reynolds numbers. Recent calculations performed in Ref. [12] for isotropic turbulence demonstrated that the total (hydrodynamic plus magnetic) α effect is nonlinearized in the form of a quenching, i.e., by replacing α with $\alpha\tilde{\Phi}(B)$, where $\tilde{\Phi}(B)$ is a decreasing function of the mean magnetic field. Note, however, that in real astrophysical applications the turbulence is anisotropic. Since $\tau_h \gg \tau_0$, the back reaction of the magnetic field on the magnetic part of the α effect cannot, in general, be reduced to a simple quenching but must be described by an evolutionary differential equation (see Refs. [9–11]). Thus there are two main types of the nonlinearities for the α effect: a quenching of the total α effect in the form of an algebraic equation (see Ref. [12]) and a nonlinear evolution of the magnetic part of the α effect which is determined by a differential equation (see Refs. [9–11]). In spite of the fact the nonlinear α effect is well studied for isotropic turbulence, effects of the mean magnetic field on the turbulent diffusion and turbulent diamagnetic velocity is still poorly understood.

In the present paper a nonlinear electromotive force for an anisotropic turbulence in the case of intermediate nonlinearity is calculated, i.e., the nonlinear mean-field dependencies of the hydrodynamic and magnetic parts of the α effect, turbulent diffusion, turbulent diamagnetic, and paramagnetic velocities for an anisotropic turbulence are found. The intermediate nonlinearity implies that the mean magnetic field is not strong enough to affect the correlation time of the turbulent velocity field. In the case of isotropic turbulence the obtained results for the α effect are in agreement with those obtained in Ref. [12]. We demonstrated that the nonlinear turbulent diamagnetic and paramagnetic velocities are determined by both an inhomogeneity of the turbulence and an inhomogeneity of the mean magnetic field **B**. The latter implies that there are additional terms in the turbulent diamagnetic and paramagnetic velocities $\propto \boldsymbol{\nabla} B^2$ and $\propto (\mathbf{B}\cdot\boldsymbol{\nabla})\mathbf{B}$. These effects are caused by a tangling of a nonuniform mean magnetic field by hydrodynamic fluctuations.

II. GOVERNING EQUATIONS

In this section we derive an equation for the nonlinear turbulent electromotive force for large hydrodynamic ($\mathrm{Re} = l_0 u_0/\nu \gg 1$) and magnetic ($\mathrm{Rm} = l_0 u_0/\eta \gg 1$) Reynolds numbers, where ν is the kinematic viscosity. We will use a mean field approach in which the magnetic, **H**, and velocity, **v**, fields are divided into the mean and fluctuating parts: $\mathbf{H} = \mathbf{B} + \mathbf{h}$, $\mathbf{v} = \mathbf{V} + \mathbf{u}$, where the fluctuating parts have zero mean values, $\mathbf{V} = \langle \mathbf{v} \rangle = \mathrm{const}$, and $\mathbf{B} = \langle \mathbf{H} \rangle$. The momentum equation and the induction equation for the turbulent fields **u** and **h** in a frame moving with a local velocity of the large-scale flows **V** are given by

$$\partial \mathbf{u}/\partial t = -\boldsymbol{\nabla} P'/\rho - [\mathbf{h}\times(\boldsymbol{\nabla}\times\mathbf{B}) + \mathbf{B}\times(\boldsymbol{\nabla}\times\mathbf{h})]/(\mu_0\rho) + \mathbf{T} + \nu\Delta\mathbf{u} + \mathbf{F}_r/\rho, \tag{3}$$

$$\partial \mathbf{h}/\partial t = \boldsymbol{\nabla}\times(\mathbf{u}\times\mathbf{B} - \eta\boldsymbol{\nabla}\times\mathbf{h}) + \mathbf{G}, \tag{4}$$

and $\boldsymbol{\nabla}\cdot\mathbf{u} = 0$, where P' are the fluctuations of the hydrodynamic pressure, $\mathbf{F}_r$ is a random external stirring force, ρ is the density of fluid, the nonlinear terms (**T** and **G**) are given by $\mathbf{T} = \langle(\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u}\rangle - (\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u} + [\langle \mathbf{h}\times(\boldsymbol{\nabla}\times\mathbf{h})\rangle - \mathbf{h}\times(\boldsymbol{\nabla}\times\mathbf{h})]/(\mu_0\rho)$, and $\mathbf{G} = \boldsymbol{\nabla}\times(\mathbf{u}\times\mathbf{h} - \langle\mathbf{u}\times\mathbf{h}\rangle)$. The fluctuations are concentrated in small scales. Therefore the derivatives of the large-scale fields are small in comparison with the derivatives of the turbulent fields. Now let us derive equations for the second moments. For this purpose we rewrite Eqs. (3) and (4) in a Fourier space and repeat twice the vector multiplication of Eq. (3) by the wave vector **k**. The result is given by

$$du_m(\mathbf{k},t)/dt = [2P_{mi}(k) - \delta_{mi}]\hat{S}_i^{(c)}(h;B)/(\mu_0\rho) + \hat{S}_m^{(b)}(h;B)/(\mu_0\rho) - \tilde{T}_m - \nu k^2 u_m, \tag{5}$$

$$dh_n(\mathbf{k},t)/dt = \hat{S}_n^{(b)}(u;B) - \hat{S}_n^{(c)}(u;B) + G_n - \eta k^2 h_n, \tag{6}$$

where $\hat{S}_n^{(c)}(a;A) = i\int a_j(\mathbf{k}-\mathbf{Q})Q_j A_n(\mathbf{Q})d^3Q$, $\hat{S}_n^{(b)}(a;A) = ik_j\int a_n(\mathbf{k}-\mathbf{Q})A_j(\mathbf{Q})d^3Q$, $\tilde{\mathbf{T}} = \mathbf{k}\times(\mathbf{k}\times\mathbf{T})/k^2$, $P_{ij}(k) = \delta_{ij} - k_{ij}$, δ_{mn} is the Kronecker tensor and $k_{ij} = k_i k_j/k^2$. We use the two-scale approach, i.e., a correlation function

$$\langle u_i(\mathbf{x})u_j(\mathbf{y})\rangle = \int \langle u_i(\mathbf{k}_1)u_j(\mathbf{k}_2)\rangle \times \exp\{i(\mathbf{k}_1\cdot\mathbf{x} + \mathbf{k}_2\cdot\mathbf{y})\}d^3k_1\, d^3k_2 = \int f_{ij}(\mathbf{k},\mathbf{R})\exp(i\mathbf{k}\cdot\mathbf{r})d^3k,$$

$$f_{ij}(\mathbf{k},\mathbf{R}) = \int \langle u_i(\mathbf{k}+\mathbf{K}/2)u_j(-\mathbf{k}+\mathbf{K}/2)\rangle \exp(i\mathbf{K}\cdot\mathbf{R})d^3K,$$

where $\mathbf{R} = (\mathbf{x}+\mathbf{y})/2$, $\mathbf{r} = \mathbf{x}-\mathbf{y}$, $\mathbf{K} = \mathbf{k}_1 + \mathbf{k}_2$, $\mathbf{k} = (\mathbf{k}_1 - \mathbf{k}_2)/2$, **R** and **K** correspond to the large scales, and **r** and **k** to the small ones (see, e.g., Refs. [13,14]). The others second moments have the same form, e.g.,

$$h_{ij}(\mathbf{k},\mathbf{R}) = \int \langle h_i(\mathbf{k}+\mathbf{K}/2)h_j(-\mathbf{k}+\mathbf{K}/2)\rangle \times \exp(i\mathbf{K}\cdot\mathbf{R})d^3K/\mu_0\rho,$$

$$\kappa_{ij}(\mathbf{k},\mathbf{R}) = \int \langle h_i(\mathbf{k}+\mathbf{K}/2)u_j(-\mathbf{k}+\mathbf{K}/2)\rangle \exp(i\mathbf{K}\cdot\mathbf{R})d^3K.$$

Note that the two-scale approach is valid when $(1/B)(dB/dR) \ll l_0^{-1}$, where $B = |\mathbf{B}|$. Now we derive the equations for the correlation functions $f_{nm}(\mathbf{k},\mathbf{R})$, $h_{nm}(\mathbf{k},\mathbf{R})$, and $\kappa_{nm}(\mathbf{k},\mathbf{R})$

$$\partial f_{nm}/\partial t = i(\mathbf{k}\cdot\mathbf{B})\Phi_{nm} + M_{nm} + F_{nm} - 2\nu k^2 f_{nm}, \tag{7}$$

$$\partial h_{nm}/\partial t = -i(\mathbf{k}\cdot\mathbf{B})\Phi_{nm} + R_{nm} - 2\eta k^2 h_{nm}, \tag{8}$$

$$\partial \kappa_{nm}/\partial t = I_{nm} + C_{nm} - (\nu+\eta)k^2\kappa_{nm}, \tag{9}$$

$$I_{nm} = i(\mathbf{k}\cdot\mathbf{B})(f_{nm} - h_{nm}) + (1/2)(\mathbf{B}\cdot\boldsymbol{\nabla}^{(R)})(f_{nm} + h_{nm}) - f_{jm}B_{nj} + h_{nj}(2P_{mi}(k) - \delta_{mi})B_{ij}, \tag{10}$$

where

$$F_{nm}(\mathbf{k},\mathbf{R}) = \langle \tilde{F}_n(\mathbf{k},\mathbf{R})u_m(-\mathbf{k},\mathbf{R})\rangle + \langle u_n(\mathbf{k},\mathbf{R})\tilde{F}_m(-\mathbf{k},\mathbf{R})\rangle,$$

$$\Phi_{nm}(\mathbf{k},\mathbf{R}) = [\kappa_{nm}(\mathbf{k},\mathbf{R}) - \kappa_{mn}(-\mathbf{k},\mathbf{R})]/\mu_0\rho,$$

$$\tilde{\mathbf{F}}(\mathbf{k},\mathbf{R},t) = \mathbf{k}\times[\mathbf{k}\times\mathbf{F}_r(\mathbf{k},\mathbf{R})]/k^2\rho,$$

and $B_{ij} = \partial B_i/\partial R_j$. The third moment is given by $M_{nm}(\mathbf{k},\mathbf{R},t) = \langle \tilde{T}_n(\mathbf{k})u_m(-\mathbf{k})\rangle + \langle u_n(\mathbf{k})\tilde{T}_m(-\mathbf{k},t)\rangle$. The expressions for the remaining moments R_{nm} and C_{nm} are similar. In Eqs. (7) and (8) we neglected the terms $\propto (\mathbf{B}\cdot\boldsymbol{\nabla}^{(R)})\kappa_{nm}$ and $\propto B_{nj}\kappa_{jm}$ because they contribute to the modification of the Ampère force caused by the turbulence effect (see, e.g., Refs. [18,19]), where $\boldsymbol{\nabla}^{(R)} = \partial/\partial\mathbf{R}$. In Eq. (9) we neglected the second and higher derivatives over **R** and terms which are of the order of $\mathrm{Rm}^{-1}\boldsymbol{\nabla}^{(R)}(B_i, f_{nm}, h_{nm})$ and $\mathrm{Re}^{-1}\boldsymbol{\nabla}^{(R)}(B_i; f_{nm}; h_{nm})$.

Now we split all correlation functions (i.e., $f_{nm}, h_{nm}, \kappa_{nm}, \Phi_{nm}$) into two parts, e.g., $f_{nm} = f_{nm}^{(N)} + f_{nm}^{(S)}$, where $f_{nm}^{(N)} = [f_{nm}(\mathbf{k},\mathbf{R}) + f_{nm}(-\mathbf{k},\mathbf{R})]/2$ and $f_{nm}^{(S)} = [f_{nm}(\mathbf{k},\mathbf{R}) - f_{nm}(-\mathbf{k},\mathbf{R})]/2$. The tensor $f_{nm}^{(S)}$ describes the helical part of the tensor, whereas $f_{nm}^{(N)}$ describes the nonhelical part of the tensor. Such splitting is caused, e.g., by different times of evolution of the helical and nonhelical parts of the magnetic tensor. In particular, the characteristic time of evolution of the tensor $h_{nm}^{(N)}$ is of the order τ_0 while the relaxation time of the component $h_{nm}^{(S)}$ is of the order of $\tau_0\,\mathrm{Rm}$ (see, e.g., Ref. [11]). By means of Eqs. (7)–(9) we derive equations for the helical and nonhelical parts of these tensors. We assume also that the magnetic tensor $h_{nm}^{(S)}$ is a given and is determined by some evolutionary equation (see Sec. V and Refs. [10,11]).

III. ELECTROMOTIVE FORCE

Equations (7)–(9) describe an evolution of the second moments. Equations of this type raise, as usual, a question of closing the equations for the higher moments. Various approximate methods have been proposed for the solution of problems of this type (see, e.g., Refs. [15–17]). The simplest closure procedure is the τ approximation, which is widely used in the theory of kinetic equations. For magnetohydrodynamic turbulence this approximation was used in Ref. [7] (see also Refs. [18,19]). In the simplest variant, it allows us to express the third moments in terms of the second moments:

$$M_{nm}-M_{nm}^{(0)}=-\frac{f_{nm}-f_{nm}^{(0)}}{\tau(k)},\quad R_{nm}^{(N)}-R_{nm}^{(0)}=-\frac{h_{nm}^{(N)}-h_{nm}^{(0)}}{\tau(k)},$$

$$C_{nm}-C_{nm}^{(0)}=-\frac{\kappa_{nm}-\kappa_{nm}^{(0)}}{\tau(k)}.$$

The superscript (0) corresponds here to the background magnetohydrodynamic turbulence (it is a turbulence with $\mathbf{B}=0$), and $\tau(k)$ is the characteristic relaxation time of the statistical moments.

The τ approximation is in general similar to the eddy damped quasinormal Markowian (EDQNM) approximation. However, some principal difference exists between these two approaches (see Refs. [15,17]). The EDQNM closures do not relax to equilibrium, and this procedure does not describe properly the motions in the equilibrium state in contrast to the τ approximation. Within the EDQNM theory, there is no dynamically determined relaxation time, and no slightly perturbed steady state can be approached [15]. In the τ approximation, the relaxation time for small departures from equilibrium is determined by the random motions in the equilibrium state, but not by the departure from equilibrium [15]. We use the τ approximation, but not the EDQNM approximation because we consider a case with $l_0|\boldsymbol{\nabla}^{(R)}B^2|/\mu_0\ll\langle\rho u^2\rangle$. As follows from the analysis by Ref. [15] the τ approximation describes the relaxation to equilibrium state (the background turbulence) much more accurately than the EDQNM approach.

Now we assume that $\nu k^2\ll\eta k^2\ll\tau^{-1}$ for the inertial range of turbulent fluid flow. We also assume that the characteristic time of variation of the mean magnetic field $\mathbf{B}$ is substantially longer than the correlation time $\tau(k)$ for all turbulence scales. Thus, Eqs. (7)–(9) yield

$$f_{nm}^{(N)}=f_{nm}^{(0N)}+i\,\tau(\mathbf{k}\cdot\mathbf{B})\Phi_{nm}^{(S)},\tag{11}$$

$$h_{nm}^{(N)}=h_{nm}^{(0N)}-i\,\tau(\mathbf{k}\cdot\mathbf{B})\Phi_{nm}^{(S)},\tag{12}$$

$$f_{nm}^{(S)}=f_{nm}^{(0S)}+i\,\tau(\mathbf{k}\cdot\mathbf{B})\Phi_{nm}^{(N)},\tag{13}$$

$$\Phi_{nm}^{(N)}=\tau(1+\psi)^{-1}(\mu_0\rho)^{-1}\{2i(\mathbf{k}\cdot\mathbf{B})(f_{nm}^{(0S)}-h_{nm}^{(S)})+B_{mj}(f_{jn}^{(0N)}+h_{jn}^{(0N)})-B_{nj}(f_{jm}^{(0N)}+h_{jm}^{(0N)})+2B_{pj}(h_{mj}^{(N)}k_{pn}-h_{nj}^{(N)}k_{pm})\},\tag{14}$$

$$\Phi_{nm}^{(S)}=2i\,\tau(1+2\psi)^{-1}(\mu_0\rho)^{-1}(\mathbf{k}\cdot\mathbf{B})(f_{nm}^{(0N)}-h_{nm}^{(0N)})+O(B_{nm}),\tag{15}$$

where $\psi=2(\mathbf{k}\cdot\mathbf{B}\tau)^2/\mu_0\rho$, $k_{ij}=k_ik_j/k^2$, $f_{ij}^{(0N)}$ and $f_{ij}^{(0S)}$ describe the nonhelical and helical tensors of the background turbulence, and we took into account that $\kappa_{ij}(\mathbf{B}=0)=0$.

Using Eqs. (11)–(15) we calculate the electromotive force $\mathcal{E}_i(\mathbf{r}=0)=\int\mathcal{E}_i(\mathbf{k})d\mathbf{k}$, where the Fourier component $\mathcal{E}_i(\mathbf{k})=(\mu_0\rho/2)\varepsilon_{imn}\Phi_{nm}^{(N)}(\mathbf{k})$, and ε_{ijk} is the Levi-Civita tensor. The electromotive force is given by

$$\mathcal{E}_i(\mathbf{r}=0)=a_{ij}B_j+b_{ijk}B_{jk},\tag{16}$$

where

$$a_{ij}=i\int\tau(1+\psi)^{-1}\varepsilon_{imn}k_j(f_{nm}^{(0S)}-h_{nm}^{(S)})d\mathbf{k},\tag{17}$$

$$b_{ijk}=\int\tau(1+\psi)^{-1}[\varepsilon_{ijn}(f_{kn}^{(0N)}+h_{kn}^{(0N)})-2\varepsilon_{imn}k_{mj}h_{nk}^{(N)}]d\mathbf{k}.\tag{18}$$

Equations (16)–(18) allow us to calculate the electromotive force. The result is given by

$$\boldsymbol{\mathcal{E}}=\hat{\boldsymbol{\alpha}}\mathbf{B}+(\mathbf{U}+\mathbf{V}^{(N)})\times\mathbf{B}-\hat{\boldsymbol{\eta}}(\boldsymbol{\nabla}\times\mathbf{B})-\hat{\boldsymbol{\kappa}}\boldsymbol{\partial}\hat{\boldsymbol{B}}\tag{19}$$

(see Appendix A), where $\hat{\alpha}_{ij}(\mathbf{B})=(a_{ij}+a_{ji})/2$, $U_k(\mathbf{B})=\varepsilon_{kji}a_{ij}/2$, $\mathbf{V}^{(N)}(\mathbf{B})=(1/2B^2)Q(\beta)\boldsymbol{\nabla}B^2$, $\hat{\kappa}_{ijk}(\mathbf{B})=-(b_{ijk}^{(2)}+b_{ikj}^{(2)})/2$, $b_{ijk}^{(2)}=\varepsilon_{ijm}K_{mk}(\Lambda)+A_{ijk}$, the turbulent magnetic diffusion is $\hat{\boldsymbol{\eta}}(\mathbf{B})=\hat{\boldsymbol{\eta}}^{(1)}+\hat{\boldsymbol{\eta}}^{(2)}$, $\hat{\eta}_{ij}^{(1)}=P_{ij}(\beta)Q(\beta)$, $\hat{\eta}_{ij}^{(2)}=\hat{K}_{ij}(\Lambda)+(\varepsilon_{ikp}A_{jkp}+\varepsilon_{jkp}A_{ikp})/4$, $\hat{K}_{ij}(\Lambda)=(K_{pp}(\Lambda)\delta_{ij}-K_{ij}(\Lambda))/2$, $\boldsymbol{\beta}=4\mathbf{B}/(u_0\sqrt{2\mu_0\rho})$, $\beta=|\boldsymbol{\beta}|$, the tensor A_{ijk} is determined by Eq. (A5), the functions $K_{ij}(\Lambda)$ and $Q(\beta)$ are determined by Eqs. (A13) and (A18), respectively (see Appendix A). The tensor $\hat{\alpha}_{ij}(\mathbf{B})$ is given by

$$\hat{\alpha}_{ij}(\mathbf{B})=(i/2)\int\tau(1+\psi)^{-1}(\varepsilon_{imn}k_j+\varepsilon_{jmn}k_i)\times(f_{nm}^{(0S)}-h_{nm}^{(S)})d\mathbf{k}.\tag{20}$$

Now we calculate the velocity $\mathbf{U}(\mathbf{B})$. The condition $\boldsymbol{\nabla}\cdot\mathbf{u}=0$ implies that $\mathbf{k}^{(2)}\cdot\mathbf{u}(\mathbf{k}^{(2)})=0$, i.e., $k_m^{(2)}f_{nm}=0$. This yields $(-ik_m+\nabla_m^{(\mathbf{R})}/2)f_{nm}(\mathbf{k},\mathbf{R})=0$. Using the change $\mathbf{k}\to-\mathbf{k}$ in the latter equation we obtain $(ik_m+\nabla_m^{(\mathbf{R})}/2)f_{nm}(-\mathbf{k},\mathbf{R})=0$. The sum of these equations yields $ik_mf_{nm}^{(S)}=\nabla_m^{(\mathbf{R})}f_{nm}^{(N)}/2$. Similarly, for an incompressible flow we get $k_n^{(1)}f_{nm}=0$ and it yields $ik_nf_{nm}^{(S)}=-\nabla_n^{(\mathbf{R})}f_{nm}^{(N)}/2$. These equations allow us to calculate the velocity $U_k(\mathbf{B})$

$=\varepsilon_{kji}a_{ij}/2=-(1/2)\nabla_p^{(\mathbf{R})}\int\tau(1+\psi)^{-1}(f_{pk}^{(0N)}-h_{pk}^{(N)})d\mathbf{k}$. Using Eq. (A3) we obtain the velocity $U_k(\mathbf{B})=U_k^{(DM)}+\tilde{U}_k^{(PM)}$, where

$$U_k^{(DM)}(\mathbf{B})=-(1/2)\nabla_p^{(\mathbf{R})}\int\tau(1+2\psi)^{-1}f_{pk}^{(0N)}\,d\mathbf{k}. \quad (21)$$

$$\tilde{U}_k^{(PM)}(\mathbf{B})=(1/2)\nabla_p^{(\mathbf{R})}\int\tau(1+2\psi)^{-1}h_{pk}^{(0N)}\,d\mathbf{k}. \quad (22)$$

The vector $\mathbf{U}^{(DM)}(\mathbf{B})$ describes the turbulent diamagnetic velocity (see, e.g., Refs. [3,4]), whereas the vector $\tilde{\mathbf{U}}^{(PM)}(\mathbf{B})$ determines the turbulent paramagnetic velocity [20] (for isotropic turbulence the turbulent paramagnetic velocity was introduced in Ref. [21]).

IV. NONLINEAR TURBULENT TRANSPORT COEFFICIENTS

In this section we calculate the nonlinear turbulent transport coefficients, i.e., the hydrodynamic and magnetic parts of the α effect, the turbulent diamagnetic and paramagnetic velocities, and the turbulent magnetic diffusion for an anisotropic turbulence.

A. The hydrodynamic part of the α effect

We find the dependence of the hydrodynamic part of the α effect on mean magnetic field, i.e., we calculate

$$\alpha_{mn}^{(v)}(\mathbf{B})=\int\frac{\alpha_{mn}^{(v)}(0,\mathbf{k})}{1+\psi(\mathbf{B},\mathbf{k})}d\mathbf{k} \quad (23)$$

(see Sec. III), where hereafter $F_{mn}(0,\mathbf{k})\equiv F_{mn}(\mathbf{B}=0,\mathbf{k})$, $\alpha_{ij}^{(v)}(0,\mathbf{k})=(i/2)\tau(\varepsilon_{imn}k_j+\varepsilon_{jmn}k_i)f_{nm}^{(0S)}$. The tensor $\alpha_{mn}^{(v)}(0,\mathbf{k})$ can be presented in the form

$$\alpha_{mn}^{(v)}(0,\mathbf{k})=3\alpha_0^{(v)}(k)k_{mn}+(3/2)[\nu_{mp}(k)k_{pn}+\nu_{np}(k)k_{pm}], \quad (24)$$

where $\alpha_0^{(v)}=\alpha_{pp}^{(v)}/3$, the anisotropic part of the hydrodynamic α tensor $\nu_{mn}=\alpha_{mn}^{(v)}-\alpha_0^{(v)}\delta_{mn}$ has the properties $\nu_{mn}=\nu_{nm}$ and $\nu_{pp}=0$. In Eq. (24) we assumed that $\alpha_0^{(v)}(k)$ and $\nu_{mn}(k)$ are independent of the direction of the wave vector. To integrate over the angles in Eq. (23) we use an identity:

$$\int\frac{k_{mn}\sin\theta}{1+a\cos^2\theta}d\theta\,d\varphi=A_1\delta_{mn}+A_2\beta_{mn},$$

where $\beta_{ps}=\beta_p\beta_s/\beta^2$, $\beta_n=4B_n/(u_0\sqrt{2\mu_0\rho})$ and

$$A_1=\frac{2\pi}{a}\left[(a+1)\frac{\arctan(\sqrt{a})}{\sqrt{a}}-1\right],$$

$$A_2=-\frac{2\pi}{a}\left[(a+3)\frac{\arctan(\sqrt{a})}{\sqrt{a}}-3\right].$$

The function $\alpha_0^{(v)}(k)$ is determined by

$$\alpha_0^{(v)}(k)=\tau(k)\chi^{(v)}(k)/12\pi k^2,$$

where

$$\chi^{(v)}(k)=(q-1)(k/k_0)^{-q}\chi_0^{(v)}(\mathbf{R})/k_0$$

is the spectrum density of the hydrodynamic helicity, $\tau(k)=2\tau_0(k/k_0)^{1-q}$ is the momentum relaxation time. We used here the fact that the hydrodynamic helicity in the limit of very small kinematic viscosity is conserved. We also assumed for simplicity that $\nu_{mn}(k)=\alpha_0(k)\nu_{mn}(\mathbf{R})$. The integration over the wave number k in Eq. (23) can be performed analytically for $q=2(1-n^{-1})$, where n is the integer number, and $n>2$ or $n\leqslant-2$. In particular, $n=6$ corresponds to the Kolmogorov spectrum $q=5/3$, $n=4$ yields the Kraichnan-Iroshnikov spectrum $q=3/2$, and $n=-2$ describes the Batchelor model of turbulence with $q=3$. Here, e.g., we present results for the Kolmogorov spectrum. The dependence of the hydrodynamic part of the α effect on the mean magnetic field is determined by

$$\alpha_{mn}^{(v)}(\mathbf{B})=\alpha_{mn}^{(v)}(0)\Psi_4(\beta)-\delta_{mn}\alpha_{ps}^{(v)}(0)\beta_{ps}\Psi_6(\beta), \quad (25)$$

where $\tilde{F}_{mn}(0)\equiv\tilde{F}_{mn}(\mathbf{B}=0)$ and the functions $\Psi_n(\beta)$ are defined in Appendix B. In the case of small mean magnetic fields ($\beta\ll1$) the result is given by $\alpha_{mn}^{(v)}(\mathbf{B})=\alpha_{mn}^{(v)}(0)(1-4\beta^2/5)-(2/5)\delta_{mn}\alpha_{ps}^{(v)}(0)\beta_p\beta_s$. For isotropic turbulence $[\alpha_{mn}^{(v)}(0)=\delta_{mn}\alpha_0^{(v)}]$ and $\beta\ll1$ the hydrodynamic part of the α effect is given by $\alpha_{mn}^{(v)}(\mathbf{B})=\alpha_0^{(v)}\delta_{mn}(1-6\beta^2/5)$. Equation (25) for $\beta\gg1$ reads $\alpha_{mn}^{(v)}(\mathbf{B})=(3\pi/10\beta)[\alpha_{mn}^{(v)}(0)-\delta_{mn}\alpha_{ps}^{(v)}(0)\beta_{ps}]$. In the case of $\beta\gg1$ and isotropic turbulence the result is given by $\alpha_{mn}^{(v)}(\mathbf{B})=(2/\beta^2)\alpha_0^{(v)}\delta_{mn}$. The latter equation is in agreement with that obtained in Ref. [12].

B. The magnetic part of the α effect

Now we find the dependence of the magnetic part of the α effect on mean magnetic field, i.e., we calculate

$$\alpha_{ij}^{(h)}(\mathbf{B})=\int\frac{\alpha_{ij}^{(h)}(\mathbf{B},\mathbf{k})}{1+\psi(\mathbf{B},\mathbf{k})}d\mathbf{k} \quad (26)$$

(see Sec. III), where the tensor $\alpha_{ij}^{(h)}(\mathbf{B},\mathbf{k})=-(i/2)\tau(\varepsilon_{imn}k_j+\varepsilon_{jmn}k_i)h_{nm}^{(S)}$ is given by $\alpha_{ij}^{(h)}(\mathbf{B},\mathbf{k})=3\alpha_0^{(h)}(\mathbf{B})k_{ij}\delta(k-k_0)/4\pi k^2$ (see Ref. [11]), and $\alpha_0^{(h)}(\mathbf{B})=2\chi^{(h)}(\mathbf{B})/(9\eta_T\mu_0\rho)$, $\chi^{(h)}(\mathbf{B})$ is the magnetic helicity. Note that the realizability condition (see, e.g., Refs. [1,4]) results in $\alpha_{ij}^{(h)}(\mathbf{B},\mathbf{k})\propto\delta(k-k_0)$ (see Ref. [11]). The integration in Eq. (26) yields

$$\alpha_{ij}^{(h)}(\mathbf{B})=\alpha_0^{(h)}(\mathbf{B})\{\Phi(\beta)\beta_{ij}+(1/2)[3-(1+\beta^2)\Phi(\beta)]P_{ij}(\beta)\}, \quad (27)$$

where $\Phi(\beta)=(3/\beta^2)[1-\arctan(\beta)/\beta]$. Note that for the turbulent mean-field dynamo the function $\alpha_{ij}^{(h)}(\mathbf{B})B_j$ only is important, i.e., $\alpha_{ij}^{(h)}(\mathbf{B})B_j=\alpha_0^{(h)}(\mathbf{B})\Phi(\beta)B_i$. Therefore we can drop the term $\propto P_{ij}(\beta)$ in Eq. (27) and rewrite it as follows: $\alpha_{ij}^{(h)}(\mathbf{B})=\alpha_0^{(h)}(\mathbf{B})\Phi(\beta)\delta_{ij}$. The latter equation is in agreement with that derived in Ref. [12]. This equation for $\beta\ll1$ reads $\alpha_{ij}^{(h)}(\mathbf{B})=\alpha_0^{(h)}(\mathbf{B})(1-3\beta^2/5)\delta_{ij}$, and for $\beta\gg1$ it is given by $\alpha_{ij}^{(h)}(\mathbf{B})=\alpha_0^{(h)}(\mathbf{B})(3\pi/2\beta^2)\delta_{ij}$.

C. The turbulent diamagnetic velocity

Now we find the dependence of the turbulent diamagnetic velocity $\mathbf{U}^{(DM)}(\mathbf{B})$ on the mean magnetic field using Eq. (21). The result is given by

$$U_i^{(DM)}(\mathbf{B}) = -(1/2)\{\nabla_p \Lambda_{pi}^{(v)}(\sqrt{2}\beta) + [\gamma^{(v)}(\sqrt{2}\beta)/B^2] \times (\mathbf{B}\cdot\nabla)B_i\}, \tag{28}$$

where the functions $\Lambda_{ij}^{(v)}(\beta)$ and $\gamma^{(v)}(\beta)$ are determined by Eqs. (A11) and (A12), respectively. In Eq. (28) we drop terms $\propto \mathbf{B}$ since they do not contribute to the electromotive force. For isotropic background turbulence the dependence of the turbulent diamagnetic velocity on the mean magnetic field is given by

$$\mathbf{U}^{(DM)}(\mathbf{B}) = -(1/2)\{\nabla[\Psi_4(\sqrt{2}\beta)\,\eta_T^{(v)}] + \eta_T^{(v)}[\Psi_6(\sqrt{2}\beta)/B^2](\mathbf{B}\cdot\nabla)\mathbf{B}\},$$

where $\eta_T^{(v)} = \tau_0 u_0^2/3$. In the case of $\beta \ll 1$ and isotropic background turbulence the result is given by

$$\mathbf{U}^{(DM)}(\mathbf{B}) = -\frac{1}{2}\left\{\nabla\left[\eta_T^{(v)}\left(1-\frac{8}{5}\beta^2\right)\right] + \frac{32}{5}\frac{\eta_T^{(v)}}{u_0^2\mu_0\rho}(\mathbf{B}\cdot\nabla)\mathbf{B}\right\},$$

and in the limit of $\beta \gg 1$ we obtain:

$$\mathbf{U}^{(DM)}(\mathbf{B}) = -\frac{3\pi}{80}\left[2^{3/2}\nabla\left(\frac{\eta_T^{(v)}}{\beta}\right) + \frac{\eta_T^{(v)}u_0\sqrt{\mu_0\rho}}{B^3}(\mathbf{B}\cdot\nabla)\mathbf{B}\right].$$

Note that in a nonlinear case the turbulent diamagnetic velocity includes terms $\propto(\mathbf{B}\cdot\nabla)\mathbf{B}$ and $\propto\nabla B$ which depend on inhomogeneity of the mean magnetic field. These effects are caused by a tangling of a nonuniform mean magnetic field by hydrodynamic fluctuations. This increases the inhomogeneity of the mean magnetic field.

D. The turbulent paramagnetic velocity

Now we find the dependence of the turbulent paramagnetic velocity $\mathbf{U}^{(PM)} = \tilde{\mathbf{U}}^{(PM)} + \mathbf{V}^{(N)}$ on the mean magnetic field (see Sec. III). The result is given by

$$U_i^{(PM)}(\mathbf{B}) = (1/2)\{\nabla_p \Lambda_{pi}^{(h)}(\sqrt{2}\beta) + [\gamma^{(h)}(\sqrt{2}\beta)/B^2] \times (\mathbf{B}\cdot\nabla)B_i\} + (1/2B^2)Q(\beta)\nabla_i B^2, \tag{29}$$

where the functions $\Lambda_{ij}^{(h)}(\beta)$, $\gamma^{(h)}(\beta)$, and $Q(\beta)$ are determined by Eqs. (A11), (A12), and (A13), respectively. In Eq. (29) we drop terms $\propto\mathbf{B}$ since they do not contribute to the electromotive force. For isotropic background turbulence the dependence of the turbulent paramagnetic velocity on the mean magnetic field is given by

$$\mathbf{U}^{(PM)}(\mathbf{B}) = (1/2)\{\nabla[\Psi_4(\sqrt{2}\beta)\,\eta_T^{(h)}] + \eta_T^{(h)}[\Psi_6(\sqrt{2}\beta)/B^2] \times (\mathbf{B}\cdot\nabla)\mathbf{B}\} + (1/6B^2)[\Psi_6(\beta)(5\eta_T^{(v)} + 3\eta_T^{(h)}) - 2\Psi_6(\sqrt{2}\beta)(\eta_T^{(v)} - \eta_T^{(h)})]\nabla B^2,$$

$\eta_T^{(h)} = \tau_0 h_0^2/3$, and h_0 is the characteristic value of magnetic fluctuations with zero mean field. In the case of $\beta \ll 1$ and isotropic background turbulence the result is given by

$$\mathbf{U}^{(PM)}(\mathbf{B}) = \frac{1}{2}\left\{\nabla\left[\eta_T^{(h)}\left(1-\frac{8}{5}\beta^2\right)\right] + \frac{32}{5}\frac{\eta_T^{(h)}}{u_0^2\mu_0\rho}(\mathbf{B}\cdot\nabla)\mathbf{B} + \frac{16}{15u_0^2\mu_0\rho}(\eta_T^{(v)} + 7\eta_T^{(h)})\nabla B^2\right\},$$

and in the limit of $\beta \gg 1$ we obtain

$$\mathbf{U}^{(PM)}(\mathbf{B}) = \frac{3\pi}{80}\left[2^{3/2}\nabla\left(\frac{\eta_T^{(h)}}{\beta}\right) + \frac{\eta_T^{(h)}u_0\sqrt{\mu_0\rho}}{B^3}(\mathbf{B}\cdot\nabla)\mathbf{B}\right] + \frac{\pi}{4\beta}\left[\frac{3}{5}\eta_T^{(v)} + \frac{1}{\sqrt{2}}\eta_T^{(h)}\right]\frac{\nabla B^2}{B^2}.$$

Therefore, the nonlinear turbulent paramagnetic velocity is determined by both an inhomogeneity of the turbulence and an inhomogeneity of the mean magnetic field $\mathbf{B}$. The latter implies that there are additional terms in the turbulent paramagnetic velocity $\propto\nabla B^2$ and $\propto(\mathbf{B}\cdot\nabla)\mathbf{B}$. These effects are caused by a tangling of a nonuniform mean magnetic field by hydrodynamic fluctuations. This increases inhomogeneity of the mean magnetic field.

E. The turbulent magnetic diffusion

The dependence of the turbulent magnetic diffusion on mean magnetic field $\mathbf{B}$ is determined by equation

$$\hat{\eta}_{ij}(\mathbf{B}) = P_{ij}(\beta)Q(\beta) + \hat{K}_{ij}(\Lambda) + D_{ij}, \tag{30}$$

where the functions $Q(\beta)$, D_{ij}, and $\hat{K}_{ij}(\Lambda) = [K_{pp}(\Lambda)\delta_{ij} - K_{ij}(\Lambda)]/2$ are given by Eqs. (A13), (A17), and (A18), respectively.

In the case of isotropic turbulence the turbulent magnetic diffusion is given by

$$\hat{\eta}_{ij}(\mathbf{B}) = (1/3)\{[\Psi_4(\beta)\delta_{ij} + \Psi_6(\beta)P_{ij}](5\eta_T^{(v)} + 3\eta_T^{(h)}) - 2[\Psi_4(\sqrt{2}\beta)\delta_{ij} + \Psi_6(\sqrt{2}\beta)P_{ij}](\eta_T^{(v)} - \eta_T^{(h)})\} + D_{ij}.$$

In the case of $\beta \ll 1$ and isotropic background turbulence the result is given by

$$\hat{\eta}_{ij}(\mathbf{B}) = \delta_{ij}[\eta_T - (2/15)\eta_T^{(\beta)}\beta^2] - (2/15)\eta_T^{(\beta)}\beta_i\beta_j + D_{ij},$$

and in the limit of $\beta \gg 1$ we obtain

$$\hat{\eta}_{ij}(\mathbf{B}) = (\pi/5\beta)\{\delta_{ij}[(5-\sqrt{2})\eta_T^{(v)} + (3+\sqrt{2})\eta_T^{(h)}] - [(3/2)\eta_T^{(v)} + (5/2^{3/2})\eta_T^{(h)}]\beta_{ij}\},$$

where $\eta_T = \eta_T^{(v)} + (5/3)\eta_T^{(h)}$, and $\eta_T^{(\beta)} = \eta_T^{(v)} + 7\eta_T^{(h)}$.

V. DISCUSSION

In this study the nonlinear mean-field dependencies of the hydrodynamic and magnetic parts of the α effect, turbulent diffusion, turbulent diamagnetic, and paramagnetic velocities for an anisotropic turbulence are found. Now we will apply the obtained results to magnetic dynamo. We first will discuss two types of nonlinearities in magnetic dynamo determined by algebraic and differential equations. Then we will derive a nonlinear system of equations for axisymmetric $\alpha\Omega$ dynamos in both spherical and cylindrical coordinates. Spherical $\alpha\Omega$ dynamo may be of relevance in convective zones of the Sun and solar type stars. The $\alpha\Omega$ dynamo in cylindrical geometry may be of relevance in galaxies.

A. Dynamic and algebraic nonlinearities

We start with a dynamic nonlinearity. To this purpose, we derive a differential equation for the magnetic part of α effect for an anisotropic turbulence. The induction equation for the magnetic field $\mathbf{H}$ is given by

$$\partial \mathbf{H}/\partial t = \nabla \times (\mathbf{v}\times\mathbf{H} - \eta \nabla\times\mathbf{H}). \tag{31}$$

The equation for the vector potential $\mathbf{A}^{(t)}$ follows from the induction equation (31)

$$\partial \mathbf{A}^{(t)}/\partial t = \mathbf{v}\times\mathbf{H} - \eta\nabla\times(\nabla\times\mathbf{A}^{(t)}) + \nabla\phi, \tag{32}$$

where $\mathbf{H}=\nabla\times\mathbf{A}^{(t)}$, $\mathbf{A}^{(t)}=\mathbf{A}+\mathbf{a}$, and $\mathbf{A}=\langle\mathbf{A}^{(t)}\rangle$ is the mean vector potential, and ϕ is an arbitrary scalar function. Now we multiply Eq. (31) by $\mathbf{a}$ and Eq. (32) by $\mathbf{h}$, add them and average over the ensemble of turbulent fields. This yields an equation for the magnetic helicity $\chi^{(h)}=\langle a_p(\mathbf{x})h_p(\mathbf{x})\rangle$:

$$\partial\chi^{(h)}/\partial t = -2\boldsymbol{\mathcal{E}}(\mathbf{B})\cdot\mathbf{B} - \chi^{(h)}/T - \nabla\cdot\mathbf{F}_\chi \tag{33}$$

(see Ref. [11]), where $\boldsymbol{\mathcal{E}}(\mathbf{B})=\langle\mathbf{u}\times\mathbf{h}\rangle$ is the electromotive force, $\chi^{(h)}/T=2\eta\langle\mathbf{h}\cdot(\nabla\times\mathbf{h})\rangle$, $T\sim\tau_0\mathrm{Rm}$ is the characteristic time of relaxation of the magnetic helicity, and $\mathbf{F}_\chi=\mathbf{V}^{\mathrm{eff}}\chi^{(h)}$ is the flux of the magnetic helicity. In the case of one preferential direction (say, in the direction $\tilde{\mathbf{e}}$) the effective velocity $\mathbf{V}^{\mathrm{eff}}=23\mathbf{V}/30+7(\tilde{\mathbf{e}}\cdot\mathbf{V})\tilde{\mathbf{e}}/10-7(\tilde{\mathbf{e}}\times\mathbf{D})/15$, and $D_i=\alpha_{ij}^{(v)}\tilde{e}_j$ (see Ref. [11]). The magnetic part of the α tensor is given by $\alpha_{ij}^{(h)}(\mathbf{B})=\alpha_0^{(h)}\Phi(\beta)\delta_{ij}$ (see Sec. IV B), where $\alpha_0^{(h)}=2\chi^{(h)}(\mathbf{B})/(9\eta_T\mu_0\rho)$. Thus, the differential equation for $\alpha_0^{(h)}$ reads

$$\frac{\partial\alpha_0^{(h)}}{\partial t}+\frac{\alpha_0^{(h)}}{T}+\nabla\cdot(\mathbf{V}^{\mathrm{eff}}\alpha_0^{(h)}) = -\frac{4}{9\eta_T\mu_0\rho}\boldsymbol{\mathcal{E}}(\mathbf{B})\cdot\mathbf{B}. \tag{34}$$

The dynamics of the magnetic part of the α effect depends on the nonlinear electromotive force $\boldsymbol{\mathcal{E}}(\mathbf{B})$ which is determined by the algebraic equation (19). Indeed,

$$\boldsymbol{\mathcal{E}}(\mathbf{B})\cdot\mathbf{B} = [\alpha_{ij}^{(v)}(\mathbf{B})+\alpha_0^{(h)}\Phi(\beta)\delta_{ij}]B_iB_j - [\hat{\boldsymbol{\eta}}(\nabla\times\mathbf{B})+\hat{\boldsymbol{\kappa}}\boldsymbol{\partial}\hat{\boldsymbol{B}}]\cdot\mathbf{B}. \tag{35}$$

Therefore, the nonlinearity in turbulent mean-field dynamo includes both nonlinearities which are determined by algebraic equation (35) and dynamic equation (34).

B. Nonlinear axisymmetric $\alpha\Omega-$ dynamo

Using results presented in Sec. IV we derive equations for a nonlinear axisymmetric $\alpha\Omega-$ dynamo for spherical and cylindrical coordinates. The mean magnetic field can be written in the form $\mathbf{B}=\mathbf{B}_p+\mathbf{B}_t$, where $\mathbf{B}_p=\nabla\times A(t,r,\theta)\mathbf{e}_\varphi$ is the poloidal field and $\mathbf{B}_t=B(t,r,\theta)\mathbf{e}_\varphi$ is the toroidal field. The nonlinear mean-field equations are given by

$$\frac{\partial}{\partial t}\begin{pmatrix}A\\B\end{pmatrix}=(\hat{L}+\hat{N})\begin{pmatrix}A\\B\end{pmatrix}, \tag{36}$$

where r,θ,φ are the spherical coordinates, the angle θ is measured from the direction of the angular velocity $\boldsymbol{\Omega}$ and

$$\hat{L}=\begin{pmatrix}\Delta_s & \alpha_{\varphi\varphi}^{(v)}(r,\theta)\\ D\hat{\Omega} & \Delta_s\end{pmatrix},\quad \hat{\Omega}A=\frac{1}{r}\frac{\partial(\Omega,Ar\sin\theta)}{\partial(r,\theta)},$$

$$\hat{N}=\begin{pmatrix}\hat{U}_1+(\zeta_1-1)\Delta_s & \alpha_{\varphi\varphi}^{(v)}\Phi_1+\alpha_{\varphi\varphi}^{(h)}\Phi\\ 0 & \hat{U}+(\zeta_1-1)\Delta_s\end{pmatrix},$$

and $\Delta_s=\Delta-1/r^2\sin^2\theta$, $\hat{U}_1=-(\mathbf{W}\cdot\nabla)r\sin\theta$, $\hat{U}=\mathbf{W}\cdot\mathbf{e}_\perp-\mathbf{W}_1\cdot\nabla$, $\mathbf{W}=\zeta_3(\beta)\mathbf{e}_\perp/r^2\sin^2\theta$, $\mathbf{W}_1=r\sin\theta\mathbf{W}$, $\zeta_3(\beta)=-(5/3)\Psi_6(\beta)+(7/6)\Psi_6(\sqrt{2}\beta)$, $\zeta_2(\beta)=(5/3)\Psi_6(\beta)-(2/3)\Psi_6(\sqrt{2}\beta)$, $\zeta_1(\beta)=(5/3)\Psi_4(\beta)-(2/3)\Psi_4(\sqrt{2}\beta)$, $\Phi_1=\Psi_4(\beta)-\Psi_6(\beta)-1$, $\mathbf{e}_\perp=\mathbf{e}_r\sin\theta+\mathbf{e}_\theta\cos\theta$ and the function $\Phi(\beta)=(3/\beta^2)[1-\arctan(\beta)/\beta]$. Equations (34) and (35) read

$$\frac{\partial\alpha_{\varphi\varphi}^{(h)}}{\partial t}+\frac{\alpha_{\varphi\varphi}^{(h)}}{T}=-B\frac{\partial A}{\partial t}+\zeta_1(\beta)\hat{M}(B,A), \tag{37}$$

where $\alpha_{\varphi\varphi}^{(h)}(r,\theta)=-\alpha_{\varphi\varphi}^{(h)}(r,\pi-\theta)$ and $\hat{M}(B,A)=(1/r^2\sin^2\theta)[\nabla(rA\sin\theta)]\cdot[\nabla(rB\sin\theta)]$. In Eqs. (36) and (37) the coordinate r and time t are measured in the units R_* and R_*^2/η_T; the $\alpha_{\varphi\varphi}^{(h)}$ is measured in the units α_*; the angular velocity Ω is measured in the units Ω_*; the vector potential of the poloidal field A and the toroidal magnetic field B are measured in units of $R_\alpha R_* B_*$ and B_*, where $R_\alpha=\alpha_* R_*/\eta_T$, $D=R_\alpha R_\Omega$ is the dynamo number, $R_\Omega=\Omega_* R_*^2/\eta_T$, R_* is the radius of a star, and $B_*=(\rho\mu_0)^{1/2}(\eta_T/R_*)$. Since we consider $\alpha\Omega-$ dynamos the terms $\sim O(R_\alpha/R_\Omega)$ are dropped in Eqs. (36) and (37), and the component $\alpha_{\varphi\varphi}$ of the tensor $\boldsymbol{\alpha}$ is only essential.

In cylindrical coordinates z,R,φ, Eqs. (36) and (37) are valid after the change $r\sin\theta\to R$, and $\hat{M}(B,A)=(1/R^2)[\nabla(RA)]\cdot[\nabla(RB)]$, $\Delta_s=\Delta-1/R^2$, $\hat{\Omega}A=\partial(\Omega,AR)/\partial(z,R)$ and R_* is the thickness of a disk, $\mathbf{e}_\perp=\mathbf{e}_R$ and $\mathbf{e}_z=\boldsymbol{\Omega}/\Omega$.

Equations (36) and (37) describe a closed nonlinear system including the algebraic and dynamic nonlinearities. For simplicity we assumed that the nonhelical part of turbulence is isotropic. We also assumed that $\nabla\eta_T=0$, i.e., homogeneous turbulent diffusion. Note that the case of pure dynamic nonlinearity was studied analytically in one-mode approximation for axisymmetric $\alpha\Omega-$ dynamo in Ref. [10], whereby a formula for the magnitude of the mean magnetic field as a function of the angular velocity and parameters for a solar-type convective zone was derived. Numerically, the case of the pure dynamic nonlinearity was studied in Ref. [22]. A complicated dynamics including appearance of a chaotic behavior of mean magnetic field was found in Ref. [22]. Analytical and numerical analysis of Eqs. (36) and (37) are subjects of future studies. Equations (36) and (37) can be generalized to the cases of nonaxisymmetric $\alpha\Omega-$ and $\alpha^2\Omega-$ dynamos.

Now we discuss a form of the electromotive force in the general case. In the isotropic case for $\beta \gg 1$ the functions $\alpha_{ij}^{(v)}(\mathbf{B})$ and $\alpha_{ij}^{(h)}(\mathbf{B})$ are proportional to B^{-2} , whereas all other turbulent transport coefficients, $\eta_{ij}(\mathbf{B})$, $\mathbf{U}(\mathbf{B})$ are proportional to B^{-1}. This implies that the growth of the mean magnetic field can be saturated only by algebraic nonlinearity. On the other hand, for anisotropic turbulence all turbulent transport coefficients including the hydrodynamic part of the α effect are proportional to B^{-1} for $\beta \gg 1$. This implies that the mean magnetic field cannot be saturated by algebraic nonlinearity alone. However, a combination of two types of nonlinearities (algebraic and dynamic) can result in a saturation of the mean magnetic field in an anisotropic case.

Note that in most astrophysical applications the condition $\partial\alpha^{(h)}/\partial t = 0$ is not valid because the relaxation time of the magnetic part of the α effect is very long, i.e., $T \sim \tau_0\,\mathrm{Rm}$. For instance, this time for galaxies is larger than the lifetime of the Universe. This implies that the nonstationary equation (34) for the magnetic part of the α effect should be solved. Note also that there are two different cases: (i) with zero mean magnetic field ($\mathbf{B}=0$) and (ii) with a small mean magnetic field. When $\mathbf{B}=\mathbf{0}$ the magnetic helicity (and the magnetic part of the α effect) is very small $\chi^{(h)} \propto \mathrm{Rm}^{-13/10}$ (see Ref. [23]). On the other hand, even for very small mean magnetic field the magnetic helicity is not small (it is of the order of the hydrodynamic helicity).

Thus, in this study a nonlinear electromotive force for an anisotropic turbulence in the case of intermediate nonlinearity is calculated. The intermediate nonlinearity implies that the mean magnetic field is not strong enough to affect the correlation time of turbulent velocity field. The case of a strong nonlinearity is a subject of future study.

APPENDIX A: DERIVATION OF EQ. (19) FOR THE ELECTROMOTIVE FORCE

In order to calculate the integral in Eq. (18) we use an identity

$$\left[\int h_{ij}(k) k_{kl} \tau(k) \exp(i\mathbf{k}\cdot\mathbf{r})\, d\mathbf{k}\right]_{r\to 0} = (4\pi)^{-1} \int \frac{1}{r} \frac{\partial^2}{\partial r_k\, \partial r_l} b_{ij}(\mathbf{r})\, d\mathbf{r} \tag{A1}$$

(see Ref. [20]), where $b_{ij}(\mathbf{r}) = \int h_{ij}(k)\tau(k)\exp(i\mathbf{k}\cdot\mathbf{r})\, d\mathbf{k}$, and we took into account that the Fourier transformation yields $ik_j \to \partial/\partial r_j$, $-\phi/k^2 \to \Delta^{-1}\phi \equiv (4\pi)^{-1}\int(\phi/r)d\mathbf{r}$. Integration in parts twice in Eq. (A1) yields

$$\int \frac{1}{r} \frac{\partial^2}{\partial r_k\, \partial r_l} b_{ij}(\mathbf{r})\, d\mathbf{r} = -\frac{4\pi}{3} b_{ij}(\mathbf{r}=0)\,\delta_{kl} - \int \frac{1}{r^3} b_{ij}(\mathbf{r})(\delta_{kl} - 3r_{kl})\, d\mathbf{r} \tag{A2}$$

(see Ref. [20]), where we used an identity: $I_{ijnm}(l=3) = -(4\pi/3) b_{nj}(\mathbf{r}=0)\delta_{im}$, and $I_{ijnm}(l<3)=0$, where $I_{ijnm}(l) = [\int (r_i b_{nj}(\mathbf{r})/r^l)\, d\sigma_m]_{r\to 0}$, and the integration in the latter integral is performed over the closed surface with an internal normal. Now we take into account that

$$h_{nm}^{(N)} = h_{nm}^{(0N)} + \psi(1+2\psi)^{-1}(f_{nm}^{(0N)} - h_{nm}^{(0N)}) \tag{A3}$$

[see Eqs. (12) and (15)]. This yields

$$b_{ijk} = \varepsilon_{ijm} K_{mk}(\lambda) + A_{ijk}, \tag{A4}$$

$$A_{ijk} = (2\pi)^{-1}\varepsilon_{kmn}\int [s_{nj}(\mathbf{r}) - s_{nj}(\mathbf{r}=0)] \times (\delta_{mi} - 3r_{mi}) r^{-3}\, d\mathbf{r}, \tag{A5}$$

$$s_{ij}(\mathbf{r}) = \lambda_{ij}^{(v)}(\mathbf{r},\beta) + \lambda_{ij}^{(h)}(\mathbf{r},\sqrt{2}\beta) - \lambda_{ij}^{(v)}(\mathbf{r},\sqrt{2}\beta), \tag{A6}$$

$$\lambda_{ij}^{(a)}(\mathbf{r},\beta) = \int \frac{c_{ij}(\mathbf{k})\tau(k)}{1+\psi(\beta,\mathbf{k})} \exp(i\mathbf{k}\cdot\mathbf{r})\, d\mathbf{k}, \tag{A7}$$

$$K_{ij}(\lambda) = [5\lambda_{ij}^{(v)}(\beta) - 2\lambda_{ij}^{(v)}(\sqrt{2}\beta) + 3\lambda_{ij}^{(h)}(\beta) + 2\lambda_{ij}^{(h)}(\sqrt{2}\beta)]/3, \tag{A8}$$

where $\beta_n = 4B_n/(u_0\sqrt{2\mu_0\rho})$, $\psi(\beta,\mathbf{k}) = [(\boldsymbol{\beta}\cdot\mathbf{k})u_0\tau/2]^2$, $\lambda_{ij}^{(a)}(\beta) = \lambda_{ij}^{(a)}(\mathbf{r}=0,\beta)$, $c_{ij} = f_{ij}^{(0N)}$ when $a=v$, and $c_{ij} = h_{ij}^{(0N)}$ when $a=h$. For the calculation of the tensor b_{ijk} we have to specify a model of the background turbulence (i.e., turbulence with zero mean magnetic field). We use the following model for the background anisotropic incompressible turbulent velocity and magnetic fields:

$$\tau c_{ij}(\mathbf{k}) = \frac{5}{4}\Bigg[P_{ij}(k)\left(\frac{2\tilde{\eta}_T^{(a)}(\mathbf{k})}{5} - \mu_{mn}^{(a)}(\mathbf{k})k_{nm}\right) + 2[\delta_{ij}\mu_{mn}^{(a)}(\mathbf{k})k_{nm} + \mu_{ij}^{(a)}(\mathbf{k}) - \mu_{im}^{(a)}(\mathbf{k})k_{mj} - k_{im}\mu_{mj}^{(a)}(\mathbf{k})]\Bigg], \tag{A9}$$

where the anisotropic part of this tensor $\mu_{mn}^{(a)}(\mathbf{k})$ has the properties $\mu_{mn}^{(a)}(\mathbf{k}) = \mu_{nm}^{(a)}(\mathbf{k})$ and $\mu_{pp}^{(a)}(\mathbf{k}) = 0$. Inhomogeneity of the background turbulence is assumed to be weak, i.e., in Eq. (A9) we dropped terms $\sim O(\boldsymbol{\nabla}(\eta_T^{(a)};\mu_{ij}^{(a)}))$. Here $\tilde{\eta}_T^{(v)}(\mathbf{k}) = \tau f_{pp}^{(0N)}(\mathbf{k}) = \eta_T^{(v)}\varphi(k)$, and $\tilde{\eta}_T^{(h)}(\mathbf{k}) = \tau h_{pp}^{(0N)}(\mathbf{k}) = \eta_T^{(h)}\varphi(k)$, $\mu_{ij}^{(a)}(k) = \mu_{ij}^{(a)}(\mathbf{R})\varphi(k)/3$, where $\varphi(k) = (\tau_1 k^2 k_0)^{-1}(k/k_0)^{-7/3}$ (see Sec. IV A). The integration in $\mathbf{k}$ space in Eq. (A7) yields

$$\lambda_{ij}^{(a)}(\beta) = \lambda_{ij}^{(a)}(\mathbf{r}=0,\beta) = \Lambda_{ij}^{(a)}(\beta) + \beta_{ij}\gamma^{(a)}(\beta), \tag{A10}$$

$$\Lambda_{ij}^{(a)}(\beta) = \Psi_1(\beta)\mu_{ij}^{(a)} + \Psi_2(\beta)(\mu_{in}^{(a)}\beta_{nj} + \beta_{in}\mu_{nj}^{(a)}) + \delta_{ij}[\eta_T^{(a)}\Psi_4(\beta) + (1/4)\Psi_3(\beta)\mu_{ps}^{(a)}\beta_{sp}], \tag{A11}$$

$$\gamma^{(a)}(\beta) = (1/2)[\Psi_5(\beta)\mu_{ps}^{(a)}\beta_{sp} + 2\Psi_6(\beta)\eta_T^{(a)}], \tag{A12}$$

where the functions $\Psi_n(\beta)$ are defined in Appendix B. For $\beta \ll 1$ the function $\gamma^{(a)}(\beta)=(2/5)\,\eta_T^{(a)}\beta^2$ and the function $\Lambda_{ij}^{(a)}(\beta)$ are given by

$$\Lambda_{ij}^{(a)}(\beta)=\delta_{ij}\left[\eta_T^{(a)}\left(1-\frac{4}{5}\beta^2\right)-\frac{8}{21}\mu_{ps}^{(a)}\beta_s\beta_p\right]+\mu_{ij}^{(a)}\left(1-\frac{22}{21}\beta^2\right)+(4/7)\beta_n(\mu_{in}^{(a)}\beta_j+\beta_i\mu_{nj}^{(a)}).$$

For $\beta \gg 1$ these functions are given by

$$\Lambda_{ij}^{(a)}(\beta)=(\pi/16\beta)\{2\mu_{ij}^{(a)}+6(\mu_{in}^{(a)}\beta_{nj}+\beta_{in}\mu_{nj}^{(a)})+\delta_{ij}[(24/5)\,\eta_T^{(a)}-5\mu_{ps}^{(a)}\beta_{sp}]\},$$

$$\gamma^{(a)}(\beta)=(3\pi/10\beta)[\eta_T^{(a)}+(5/8)\mu_{ps}^{(a)}\beta_{sp}].$$

For an isotropic turbulence $\gamma^{(a)}(\beta)=\Psi_6(\beta)\,\eta_T^{(a)}$ and $\Lambda_{ij}^{(a)}(\beta)=\delta_{ij}\Psi_4(\beta)\,\eta_T^{(a)}$. Now we calculate $\mathcal{E}_i^{(1)}=b_{ijk}^{(1)}B_{jk}$, where $b_{ijk}^{(1)}=\varepsilon_{imj}\beta_{mk}Q(\beta)$, and

$$Q(\beta)=[5\gamma^{(v)}(\beta)-2\gamma^{(v)}(\sqrt{2}\beta)+3\gamma^{(h)}(\beta)+2\gamma^{(h)}(\sqrt{2}\beta)]/3. \tag{A13}$$

For $\beta \ll 1$ the function $Q(\beta)$ is given by $Q(\beta)=(2\beta^2/15)(\eta_T^{(v)}+7\,\eta_T^{(h)})$, and for $\beta \gg 1$ the function $Q(\beta)$ is given by

$$Q(\beta)=\frac{\pi}{2^{3/2}\beta}\left(\frac{3\sqrt{2}}{5}\eta_T^{(v)}+\eta_T^{(h)}+\frac{1}{8}\beta_{sp}(\sqrt{2}\mu_{ps}^{(v)}+5\mu_{ps}^{(h)})\right). \tag{A14}$$

For an isotropic turbulence $Q(\beta)=(1/3)[\Psi_6(\beta)(5\,\eta_T^{(v)}+3\,\eta_T^{(h)})-2\Psi_6(\sqrt{2}\beta)(\eta_T^{(v)}-\eta_T^{(h)})]$. Now we use an identity $\varepsilon_{imn}\beta_{np}B_{mp}=-[\mathbf{B}\times\boldsymbol{\nabla}(B^2/2)]_i/B^2-P_{ip}(\beta)(\boldsymbol{\nabla}\times\mathbf{B})_p$, where $P_{ij}(\beta)=\delta_{ij}-\beta_{ij}$. This yields $\mathcal{E}_i^{(1)}=[\mathbf{V}^{(N)}(\mathbf{B})\times\mathbf{B}]_i-\hat{\eta}_{ij}^{(1)}(\boldsymbol{\nabla}\times\mathbf{B})_j$, where the velocity $\mathbf{V}^{(N)}(\mathbf{B})=(1/2)Q(\beta)(\boldsymbol{\nabla}B^2)/B^2$ describes an additional contribution to the nonlinear turbulent paramagnetic velocity, and $\hat{\eta}_{ij}^{(1)}=P_{ij}(\beta)Q(\beta)$ determines an additional contribution to the nonlinear turbulent diffusion. The total electromotive force is given by $\boldsymbol{\mathcal{E}}=\boldsymbol{\mathcal{E}}^{(1)}+\boldsymbol{\mathcal{E}}^{(2)}$, where $\mathcal{E}_i^{(2)}=a_{ij}B_j+b_{ijk}^{(2)}B_{jk}$ and $b_{ijk}^{(2)}=\varepsilon_{ijm}K_{mk}(\Lambda)+A_{ijk}$. Now we use an identity $B_{jk}=(\partial\hat{B})_{jk}-\varepsilon_{jkl}(\boldsymbol{\nabla}\times\mathbf{B})_l/2$ (see Ref. [24]), where $(\partial\hat{B})_{jk}=(B_{jk}+B_{kj})/2$. This yields

$$\boldsymbol{\mathcal{E}}^{(2)}=\hat{\boldsymbol{\alpha}}\mathbf{B}+\mathbf{U}\times\mathbf{B}-\hat{\boldsymbol{\eta}}^{(2)}(\boldsymbol{\nabla}\times\mathbf{B})-\hat{\boldsymbol{\kappa}}\boldsymbol{\partial}\hat{\mathbf{B}} \tag{A15}$$

where $\hat{\alpha}_{ij}(\mathbf{B})=(a_{ij}+a_{ji})/2$, $U_k(\mathbf{B})=\varepsilon_{kji}a_{ij}/2$, $\hat{\kappa}_{ijk}(\mathbf{B})=-(b_{ijk}^{(2)}+b_{ikj}^{(2)})/2$,

$$\hat{\eta}_{ij}^{(2)}=(\varepsilon_{ikp}b_{jkp}^{(2)}+\varepsilon_{jkp}b_{ikp}^{(2)})/4=\hat{K}_{ij}(\Lambda)+D_{ij}, \tag{A16}$$

$$D_{ij}=(\varepsilon_{ikp}A_{jkp}+\varepsilon_{jkp}A_{ikp})/4, \tag{A17}$$

and $\hat{K}_{ij}(\Lambda)=[K_{pp}(\Lambda)\,\delta_{ij}-K_{ij}(\Lambda)]/2$, and

$$K_{ij}(\Lambda)=[5\Lambda_{ij}^{(v)}(\beta)-2\Lambda_{ij}^{(v)}(\sqrt{2}\beta)+3\Lambda_{ij}^{(h)}(\beta)+2\Lambda_{ij}^{(h)}(\sqrt{2}\beta)]/3. \tag{A18}$$

For $\beta \ll 1$ the function $K_{ij}(\Lambda)$ is given by

$$K_{ij}(\Lambda)=\eta_{ij}(\mathbf{B}=0)+(4/7)\beta_n(\beta_j\mu_{in}+\beta_i\mu_{nj})-\frac{\beta^2}{3}\left(\frac{4}{5}\delta_{ij}\,\eta_T^{(\beta)}+\frac{22}{21}\mu_{ij}^{(\beta)}+\frac{8}{21}\delta_{ij}\beta_{sp}\mu_{ps}^{(\beta)}\right), \tag{A19}$$

where $\eta_{ij}(\mathbf{B}=0)=\delta_{ij}\,\eta_T+\mu_{ij}$, $\eta_T=\eta_T^{(v)}+(5/3)\,\eta_T^{(h)}$, $\mu_{ij}=\mu_{ij}^{(v)}+(5/3)\mu_{ij}^{(h)}$, $\eta_T^{(\beta)}=\eta_T^{(v)}+7\,\eta_T^{(h)}$, and $\mu_{ij}^{(\beta)}=\mu_{ij}^{(v)}+7\mu_{ij}^{(h)}$. For $\beta \gg 1$ the function $K_{ij}(\Lambda)$ is given by

$$K_{ij}(\Lambda)=\frac{\pi}{48\beta}[(5-\sqrt{2})\Lambda_{ij}^{(v)}(\beta)+(3+\sqrt{2})\Lambda_{ij}^{(h)}(\beta)].$$

For an isotropic turbulence

$$K_{ij}(\Lambda)=(\delta_{ij}/3)[\Psi_4(\beta)(5\,\eta_T^{(v)}+3\,\eta_T^{(h)})-2\Psi_4(\sqrt{2}\beta)(\eta_T^{(v)}-\eta_T^{(h)})].$$

Therefore the total electromotive force is given by Eq. (19).

APPENDIX B

The functions $\Psi_k(\beta)$ are given by

$$\Psi_1(\beta)=\frac{5}{4}\left[\frac{\arctan\beta}{\beta}\left(\frac{1}{5}-\frac{6}{7\beta^2}+\frac{1}{9\beta^4}\right)+\frac{92}{315}L(\beta)+\frac{169}{189\beta^2}-\frac{1}{9\beta^4}\right],$$

$$\Psi_2(\beta)=\frac{5}{4}\left[\frac{\arctan\beta}{\beta}\left(\frac{3}{5}+\frac{6}{7\beta^2}-\frac{5}{9\beta^4}\right)-\frac{64}{315}L(\beta)-\frac{197}{189\beta^2}+\frac{5}{9\beta^4}\right],$$

$$\Psi_3(\beta)=-\frac{5}{2}\left[\frac{\arctan\beta}{\beta}\left(1+\frac{18}{7\beta^2}+\frac{5}{9\beta^4}\right)-\frac{16}{63}L(\beta)-\frac{451}{189\beta^2}-\frac{5}{9\beta^4}\right],$$

$$\Psi_4(\beta)=\frac{3}{35}\left[\frac{\arctan\beta}{\beta}\left(7-\frac{5}{\beta^2}\right)+3L(\beta)+\frac{5}{\beta^2}\right],$$

$$\Psi_5(\beta)=\frac{25}{4}\left[\frac{\arctan\beta}{\beta}\left(\frac{3}{25}+\frac{6}{7\beta^2}+\frac{7}{9\beta^4}\right)+\frac{16}{1575}L(\beta)-\frac{113}{189\beta^2}-\frac{7}{9\beta^4}\right],$$

$$\Psi_6(\beta)=\frac{3}{35}\left[\frac{\arctan\beta}{\beta}\left(7+\frac{15}{\beta^2}\right)-2L(\beta)-\frac{15}{\beta^2}\right],$$

where $L(\beta)=1-2\beta^2+2\beta^4\ln(1+\beta^{-2})$. In the case of $\beta\ll 1$ these functions are given by

$$\Psi_1(\beta)\sim 1-(22/21)\beta^2,\quad \Psi_2(\beta)\sim(4/7)\beta^2,$$

$$\Psi_3(\beta)\sim -(32/21)\beta^2,$$

$$\Psi_4(\beta)\sim 1-(4/5)\beta^2,\quad \Psi_5(\beta)\sim -(16/63)\beta^4\ln\beta,$$

$$\Psi_6(\beta)\sim(2/5)\beta^2.$$

In the case of $\beta\gg 1$ these functions are given by

$$\Psi_1(\beta)\sim\pi/8\beta,\quad \Psi_2(\beta)\sim 3\pi/8\beta,\quad \Psi_3(\beta)\sim -5\pi/4\beta,$$

$$\Psi_4(\beta)\sim 3\pi/10\beta,\quad \Psi_5(\beta)\sim 3\pi/8\beta,\quad \Psi_6(\beta)\sim 3\pi/10\beta.$$

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